Research

Factor Exposure and Portfolio Concentration

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Contents

1 Introduction

Factors have been debated in the academic literature for many decades. That risk premia exist for factors such as value, momentum, size, low volatility and quality seems pretty much settled [1 - 7]. Given this, the discussion has progressed to the circumstances under which one should try to access these risk premia [9, 31] and to what portfolio construction techniques one should employ to do this efficiently.

Some researchers advocate a market neutral approach, accessing "pure" factor premia utilising long-short portfolio techniques. Others take a long only approach, viewing the premia as more efficient way of accessing market exposure. We feel this is a matter of judgment for the individual manager which will be based in part on risk appetite and the limitations imposed by their investment mandates. In this paper we will consider the long only approach although much of the material may be easily extended to create long-short portfolios.

The use of optimization [10, 11, 12] has been an important tool in the selection and weighting of stocks since the work of Markowitz [29]. Put simply one can specify preferences around factor exposures, diversification, risk etc. and then let the "black box" do the work of delivering a portfolio that satisfies these criteria. If the problem is set up correctly, and the black box does its job of efficiently finding the optimal solution, this can deliver the desired outcomes. The one drawback to optimization is around transparency. The black box "knows" why it has chosen stocks and in what proportion, but this may not be so clear to the human who allocated it the task.

Given this issue with optimization, a number of ad hoc methods of portfolio construction have been developed. The simplest and most commonly employed is the construction of a Characteristic Basket, which selects a given proportion of some initial universe by factor value. Stocks within the basket are then weighted according to the factor value itself or on some other criteria concerned with capacity (e.g. Market Cap weighting), diversification (e.g. Equal weighting) or risk (e.g. Risk weighting).

Another simple, but widely used approach is that of "factor tilting" [8, 14, 32]. The idea here is to take a starting universe of stock weightings and to perturb them in a way that increases the exposure to the factor of interest. This is often achieved by multiplying the initial set of weights by a scoring function; high scores for stocks with large factor values and close to zero scores for stocks with the smallest factor values.

One criticism of such tilting techniques, often made by advocates of a selection approach, is that it can only ever provide relatively weak factor exposure. Quite correctly, they highlight that the factor exposure of say, a Characteristic Basket, can be readily increased by further narrowing of the selection universe, for example by taking the top 10% by factor value rather than the top 50%. In this paper we show that criticism regarding factor exposure strength resulting from tilting approaches is quite wrong, and that exposure outcomes depend on the tilting function employed. Indeed, we show that the Characteristic Basket is a special case of a factor tilt, where the scoring function is the step function.

The final, and arguably the most important question is how should one construct multi-factor portfolios. Leaving aside the possibility of optimized solutions, what is the most appropriate mechanism for obtaining multiple factor exposure, whilst maintaining appropriate levels of stock weight diversification? Two ways in which this is done can be characterised as via "top down" (or mixed) portfolios and "bottom up" (or integrated) portfolios.

In the "top down" approach one constructs a composite portfolio from single factor portfolios. Stock weights in this multi-factor portfolio are a weighted average of their weights in the single factor portfolios. Examples of this are given in [13, 17, 19].

Alternatively, in a "bottom up" portfolio, a particular stock is weighted in consideration of all its factor characteristics simultaneously. An example of this is by use of a composite factor, where individual factor values are combined in some way resulting in an overall factor score that is used for stock selection and weighting.

The relative merits and drawbacks of these general approaches have been discussed extensively in the financial literature [15, 25, 26, 27, 28, 35]. Proponents of "top down" suggest it provides the greatest factor exposure consistent with a high degree of diversification [13, 16, 17]. Although it is accepted that the averaging of stock weights results in an averaging of factor exposures, "top down" advocates argue that high multi-factor exposures can be maintained by averaging high exposure single factor portfolios. However, such high exposure single factor portfolios can only be maintained through the application of increasingly aggressive stock selection and weighting, with adverse implications for levels of diversification. On the other hand some practitioners [16, 17] appear unconcerned regarding the potentially relatively weak factor exposures engendered by averaging and choose to highlight the composite portfolio's stock weight diversification benefits. This seems strange since, whilst it is clear that diversification is important, the primary target of a factor portfolio should surely be intentional factor exposure.

We are firm proponents of the bottom up approach. In this paper we concentrate on an alternative to the composite factor method described above. This is the concept of multiple factor tilting, where we apply sequential factor tilts to a given starting universe of stocks in the expectation that it will achieve both the multiple factor exposures we require and acceptable levels of stock diversification.

The objective of this paper is to assess the relative benefits and drawbacks of the various factor and multi-factor portfolio construction techniques described above, through the lens of factor exposure and portfolio diversification. Academic and empirical evidence tells us that portfolio exposure to certain factors is a good thing [1 - 7], while modern portfolio theory emphasizes the importance of diversification [34].

To avoid reliance on empirical data and criticisms that our results are sample specific, we prefer a more theoretical approach. We make the assumption that our factors can be modelled by a normal distribution and perform our calculations in the continuous limit. We are clear that whilst real portfolios do not contain infinitely many stocks; that correlated factors are not identically normally distributed and stock weights are not infinitesimal, such abstractions are common in finance, and so long as we appreciate its limitations, it can offer useful guidance for the real world.

Section 2 sets out discrete definitions of composite and multiple tilt portfolios and introduces the definitions of exposure and diversification used throughout the remainder of this paper. Section 3 extends the constructs of Section 2 to the continuous limit and derives formulae for use in later sections. In Section 4 we set out two important tilt functions that form the basis for all of our subsequent results, which are set out in Section 5, where we compare outcomes of one, two, three and N - factor portfolios employing alternative construction techniques. Section 6 concludes.

2 Factor Tilting

The concept of factor tilting has a long pedigree [8, 14, 32]. The basic idea is to start with the set of portfolio weights $\widehat{W}_i$ from which we wish to tilt and to define a set factor values f_i for each stock labelled by $i = 1, .., N$. Since the factor values involve different sets of natural units and ranges, it is convenient but not essential, to rescale and truncate these factor values to form Z-Scores according to:

$$Z_i = \frac{f_i - \mu}{\sigma} \tag{1}$$

where μ and σ are the cross sectional mean and standard deviation. Different factors can now be more readily compared since they all have mean zero and standard deviation one.

The next consideration is to apply some function $F(Z)$ that maps each of the Z-Scores to a positive real number. The functional form chosen is important, since it will determine many of the properties of the tilted portfolio. The tilted portfolio weights are then defined by:

$$W^T{}_i = \frac{F(Z_i) * \widehat{W}_i}{\sum_{j=1}^{N} F(Z_j) * \widehat{W}_j} \tag{2}$$

Tilting away from the factor is a simple matter of reversing the sign of the Z-Score.

An appreciation that the starting weights in (2) can take any form and not merely the standard market capitalisation benchmark weights, means it is a natural extension to take a set of previously factor tilted weights as a starting point. This simple consideration leads naturally to the notion of multiple factor tilting [32].

The formula for the weights for multiple factor tilt is obtained by iteratively applying (2). It has the simple form:

$$W^T{}_i = \frac{F_1(Z_{1,i}) * ... * F_n(Z_{n,i}) * \widehat{W}_i}{\sum_{j=1}^{N} F_1(Z_{1,j}) * ... * F_n(Z_{n,j}) * \widehat{W}_j} \tag{3}$$

where $Z_{m,i}$ is the Z-Score of the mth factor for the ith stock. Note that the $F_m(Z)$ need not all have the same functional form and that, since we are multiplying the functions, the order of the multiple tilting is immaterial.

The result is a multi-factor portfolio – but of course this is not the only way to create such a portfolio. The simplest and most common construction is the composite portfolio. This is created by taking a weighted average of the weights of several single factor portfolios with weights $W^{F_m}{}_i$ thus:

$$W^C{}_i = \alpha_1 * W^{F_1}{}_i + .. + \alpha_n * W^{F_n}{}_i \tag{4}$$

where the α_m are positive real numbers satisfying :

$$\sum_{m=1}^{n} \alpha_m = 1 \tag{5}$$

This is often referred to as a "top down" construction technique whereas multiple tilting would be characterised as "bottom up".

To assess how much of the factor characteristic is embedded in a given portfolio, we define the Factor Exposure as:

$$E_Z[W] = \sum_{i=1}^{N} W_i * Z_i \tag{6}$$

where W_i is the set of portfolio weights. The Active Factor Exposure, relative to another set of weights $\widehat{W}$ is defined by:

$$AE_Z[W, \widehat{W}] = E_Z[W] - E_Z[\widehat{W}] \tag{7}$$

So for example, if the tilt function has been chosen appropriately in (2), the Active Factor Exposure should be positive.

To assess the degree of diversification in portfolio, the Herfindahl measure of concentration [23] can be used:

$$D[W] = \sum_{i=1}^{N} W_i{}^2 \tag{8}$$

with the effective number or "Effective N", or diversification of stocks given by $1/D[W]$. Effective N attains its maximum under an equal weighting scheme and is equal to the actual number of stocks. Hence, Effective N can be seen as a measure of "how far" a given portfolio is from this most diversified portfolio. There are many alternative measures of diversification [18, 24], but this is one based only on weighting considerations.

In this Section we have given a general description of how to construct multi-factor portfolios based on multiple tilting and composite portfolios. We could provide examples of real world indexes using specific stock universes and time periods and compare factor exposure outcomes and levels of diversification - we could even compare their performances. However any demonstration of the superiority of the multiple tilt technique in terms of exposure and diversification would be subject to the criticism that the results are specific to a chosen set of circumstances.

In the next Section we attempt to address this issue by abstracting all of the above into to the continuous limit along with some distributional assumptions about factors. Irrespective of any debate over the practical applicability in making such a leap, we shall attempt to gain clearer insights into the properties of alternative approaches to multi-factor portfolio construction.

3 The Continuous Limit

To this point, we have made no assumptions regarding the distributional form taken by factor values or Z-Scores. In what follows, we make the assumption that each of our Z-Scores X_i follows a normal distribution with mean zero and standard deviation one:

$$N(x) = \frac{1}{\sqrt{2\pi}} \, \mathrm{Exp}\left(-\frac{1}{2}x^2\right)$$

(9)

It is clear that factors can and do take on distributions that are very different from the normal distribution. However, it is also clear that many factors do in fact approximate a bell-shaped distribution or may be transformed to a "normal like" state. For example, the size factor as represented market capitalisation will for many stock universes (particularly large ones) of interest follow approximately a log-normal type distribution [33]. A log transformation will therefore result in the desired factor distribution. It is also worth noting that it is a common practice to form composite factors, that is, factors are derived from sums of different sub-component factor Z-Scores. Often, even when the original factors are not normally distributed but more or less independent, their sum will tend to have a distribution that is closer to normality. These considerations suggest that the assumption of normality is not overly restrictive.

The correlation of factors varies. For example quality tends to be positively correlated to low volatility, whereas value tends to be negatively correlated to momentum. This is encapsulated in the factor correlation matrix $\boldsymbol{\rho}$ whose elements are defined by:

$$\rho_{ij} = \frac{\Sigma_{ij}}{\sigma_{X_i} * \sigma_{X_j}}$$

(10)

where $\Sigma_{ij} = \mathrm{Cov}(X_i, X_j)$ is the covariance matrix and σ_{X_i} is the square root of the variance of X_i. Multiple normally distributed factors are then jointly distributed according to the multivariate normal distribution:

$$P(\boldsymbol{\rho}, x_1, \ldots, x_n) = \mathrm{Exp}\left(-\frac{1}{2} \, \boldsymbol{x}^T \boldsymbol{\Sigma}^{-1} \boldsymbol{x}\right) \Big/ \sqrt{(2\pi)^n |\boldsymbol{\Sigma}|}$$

(11)

Given this distributional structure of our factors, our intention is to extend the formulae of Section 2 to the continuous limit. This will allow us to derive firm statistical results that should be valid for the discrete case where sample size is large and for averages derived from sufficiently large numbers of portfolio realizations.

In the continuous limit, we define a weight function $W(x_1, \ldots, x_n)$ as any function that satisfies:

$$\iint\limits_{-\infty}^{\infty} P(\boldsymbol{\rho}, x_1, \ldots, x_n) \, W(x_1, \ldots, x_n) \, dx_1 \ldots dx_n = 1 \quad \text{and} \quad W(x_1, \ldots, x_n) \geq 0$$

(12)

This can be viewed as the statement that weights are positive and add up to one.

Analogous to (6) we define the exposure of the factor X_i associated with this weight function by:

$$E_{X_i}[W] = \iint\limits_{-\infty}^{\infty} P(\boldsymbol{\rho}, x_1, \dots, x_n) \, W(x_1, \dots, x_n) \, x_i \, dx_1 \dots dx_n \tag{13}$$

The concentration (or the reciprocal of Effective N) follows from (8) by:

$$D[W] = \iint\limits_{-\infty}^{\infty} P(\boldsymbol{\rho}, x_1, \dots, x_n) \, W(x_1, \dots, x_n)^2 \, dx_1 \dots dx_n \tag{14}$$

Note that for equal weighting $W(x_1, \dots, x_n) = 1$ so that Effective N, given by $1/D[W]$, is equal to one. Henceforth we can refer to the Effective N measure, calculated in the continuous limit, as a percentage.

Let $F_i(x) \geq 0$ be a tilt function. In the continuous limit and starting from equal stock weighting, the equivalent equation to the discrete multiple tilting equation (3) is:

$$W^T[F_1(x_1), \dots, F_n(x_n)] = \frac{F_1(x_1) \dots F_n(x_n)}{\iint_{-\infty}^{\infty} P(\boldsymbol{\rho}, x_1, \dots, x_n) \, F_1(x_1) \dots F_n(x_n) \, dx_1 \dots dx_n} \tag{15}$$

It is clear that this expression satisfies both conditions in (12) to be a weight function. Note that since the exposure arising from equal weighting is zero in what follows exposure and active exposure are identical.

Whilst tilting from an equally weighted starting point may seem restrictive, it is worth noting that multiple tilts can be viewed as sequential. Hence, a weight function resulting from the first tilt $F_1(x_1)$ can be interpreted as a set of starting weights. So for example, tilting from equal weighting towards a suitably defined large size factor will yield a new set of starting weights that is more in line with a set of capitalization weights.

It is useful to write explicit formulae for exposure and concentration for the single factor case:

$$E_X[W^T[F(x)]] = \frac{\int_{-\infty}^{\infty} N(x) \, F(x) \, x \, dx}{\int_{-\infty}^{\infty} N(x) \, F(x) \, dx} \quad \text{and} \quad D[W^T[F(x)]] = \frac{\int_{-\infty}^{\infty} N(x) \, F(x)^2 \, dx}{\left[\int_{-\infty}^{\infty} N(x) \, F(x) \, dx\right]^2} \tag{16}$$

We can write these equations in a more compact form by defining:

$$I[F(x)] = \int_{-\infty}^{\infty} N(x) \, F(x) \, dx \tag{17}$$

The equations for exposure and concentration become:

$$E_X[W^T[F(x)]] = \frac{I[F(x) * x]}{I[F(x)]} \quad \text{and} \quad D[W^T[F(x)]] = \frac{I[F(x)^2]}{I[F(x)]^2} \tag{18}$$

For a composite portfolio of single tilted portfolios we have the following analogue to (4):

$$W^C[F_1(x_1), \dots, F_n(x_n)] = \alpha_1 * \frac{F_1(x_1)}{I(F_1)} + \dots + \alpha_n * \frac{F_n(x_n)}{I(F_n)} \tag{19}$$

where the $\alpha_m > 0$ and sum to one. It is easily checked that this function satisfies the properties of a weight function.

The expressions for the exposure and diversification of the composite portfolio (19) can be simplified using the following expression for the marginal joint probability function:

$$\int_{-\infty}^{\infty} P(\boldsymbol{\rho}, x_1, \dots, x_n)\, dx_j = P(\widehat{\boldsymbol{\rho}}, x_1, \dots x_{j-1}, x_{j+1} \dots, x_n) \tag{20}$$

where $\widehat{\boldsymbol{\rho}}$ is the correlation matrix without the jth row and column. The following equations for the marginal probability and marginal expectation of two factors are also useful:

$$\int_{-\infty}^{\infty} P(\rho_{ij},\, x_i,\, x_j)\, dx_j = N(x_i) \quad \text{and} \quad \int_{-\infty}^{\infty} P(\rho_{ij},\, x_i,\, x_j)\, x_i\, dx_i = \rho_{ij}\, x_j\, N(x_j) \tag{21}$$

Applying these identities iteratively to the expression (13) for exposure gives:

$$E_{X_i}[W^C[F_1(x_1),\dots,F_n(x_n)]\,] = \sum_{j=1}^{n} \alpha_j * \rho_{ij} * E_{X_j}[W^T[F_j(x_j)]] \tag{22}$$

That is, the exposure of factor X_i is a linear sum of its correlations with the other factors multiplied by the exposure of each of the other factors.

Substituting (19) into (14) and then iterative application of (20) and (21) gives the following expression for the concentration:

$$\begin{aligned}
D[W^C[F_1(x_1),\dots,F_n(x_n)]] \\
= \sum_{i=1}^{n} \alpha_i{}^2 * D[W^T[F_i(x_i)]] + \sum_{i,j \neq i}^{n} \frac{\alpha_i}{I[F_i]} * \frac{\alpha_j}{I[F_j]} * G[\rho_{ij}, F_i(x_i), F_j(x_j)]
\end{aligned} \tag{23}$$

where the second summation is a double sum over i and j and G is defined by:

$$G[\rho_{ij}, F_i(x_i), F_j(x_j)] = \iint_{-\infty}^{\infty} P(\rho_{ij},\, x_i, x_j)\, F_i(x_i) * F_j(x_j)\, dx_i\, dx_j \tag{24}$$

In other words the concentration of the composite portfolio is a linear sum of the single factor concentrations plus a piece that involves the sum of an integral expression involving the pairwise correlations of the factors and their tilt functions.

4 Functional Form of the Tilt Function

As we have stated the function $F(x)$ can have any functional form so long as it is not negative. In identifying an appropriate tilt function there are some general considerations. First, stocks with higher factor values should not receive a smaller weight than stocks with smaller factor values. Hence the function should be monotonic (non-decreasing). Second, one would probably want to limit the over weighting of extreme Z-Scores. Such scores tend to be unstable or even the result of data errors. This leads to functional forms that level out at large or small Z-scores, implying some form of "S"-shape curve.

Two functional forms that satisfy these criteria are of particular interest. The first is the Step Function:

$$H(x) = \begin{cases} 0 & for \quad x < S^{-1}(p) \\ 1 & for \quad x \geq S^{-1}(p) \end{cases} \tag{25}$$

where S^{-1} is the inverse of the Cumulative Normal function:

$$S(x) = \int_{-\infty}^{x} N(y)\, dy = \frac{1}{2}\left[1 + \mathrm{Erf}\left(\frac{x}{\sqrt{2}}\right)\right] \tag{26}$$

where $\mathrm{Erf}(x)$ is the error function and p is a number between zero and one. Figure 1 displays a plot of the Step Function where we have chosen p as 0.5.

Figure 1: The Step Function
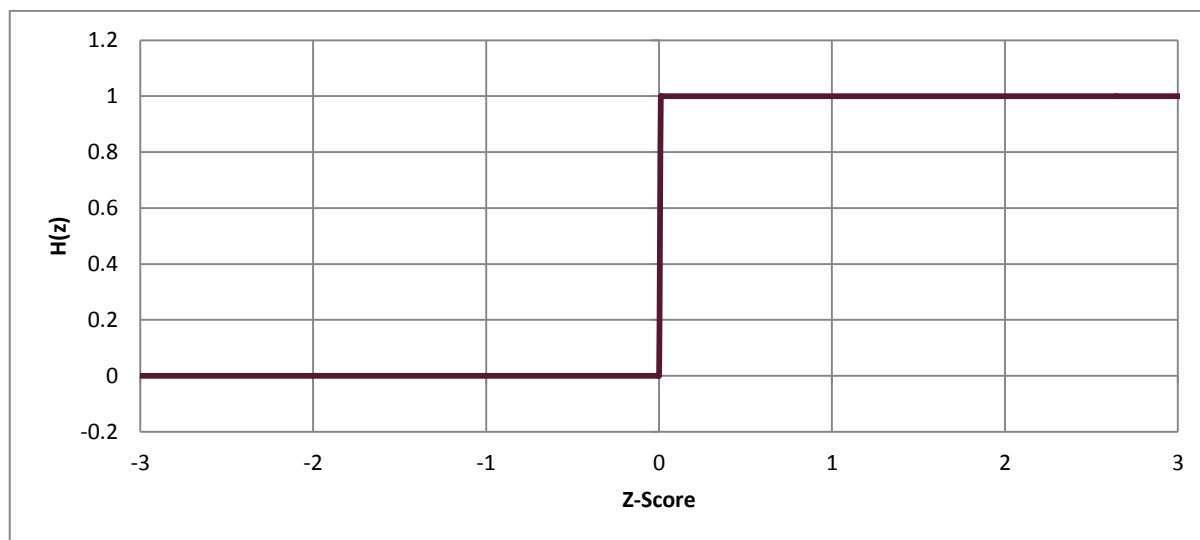
We can interpret $H(x)$ as simply the function that when applied as a tilt weights everything above the $100 * p^{th}$ percentile equally and everything below at zero. This corresponds to the classic construction methodology of forming a Characteristic Basket, consisting of a given proportion of the stock universe by factor value. Hence this classic factor portfolio construction technique can interpreted as a special case of a tilt.

The second function is the Cumulative Normal $S(x)$ itself [32]. This is particularly apt since we have hitherto made the assumption that our underlying factor values are normally distributed. This has the additional advantages that it is symmetric and is the unique tilt function that is equivalent to a simple ranking in the case of normally distributed factors. Figure 2 shows a plot of the Cumulative Normal function.

Figure 2: The Cumulative Normal
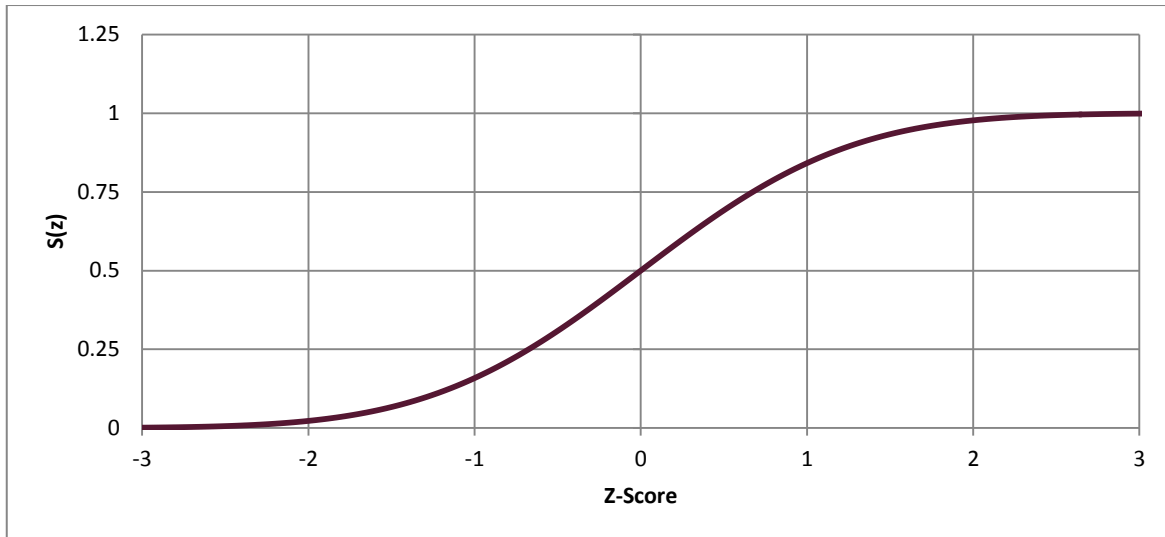
It is clear that we can apply a given factor's tilt function repeatedly to an underlying set of weights. In particular we can set $F(x) = S(x)^n$ where n can be any real positive number. This gives rise to the notion of increasing (or decreasing) the strength for the tilt. We will refer to this as the "power" of the tilt. The "flattening out at extremes" now becomes an essential property of a tilt function since, were this not the case, stocks with large positive Z-scores would rapidly dominate the portfolio as n increases, potentially leading to concentration issues.

Note that it doesn't make sense to choose the step function $H(x)$ for a given factor and then apply it repeatedly since it is clear that $H(x)^n = H(x)$. Instead the strength of tilt of the step function can be altered by varying value of the percentile parameter in (25). Higher percentile values for p will result in stronger factor tilts.

The remainder of this paper will be concerned with investigating the properties of these alternative tilt functions for use in constructing single and multi-factor portfolios. We will use the formulae discussed in the previous section to assess the exposure and diversification outcomes of these construction techniques.

5 Exposure and Concentration

In this section, we discuss important outcomes for factor portfolio construction: aggregate levels of factor exposure and the degree of diversification. We do this by looking at the exposure and diversification properties of portfolios constructed from one, two, three and then finally N factors.

We expect aggregate factor exposure and diversification to work in opposite directions. To see this, consider a portfolio consisting of a single stock that has the maximum target factor value. Now consider an equally weighted portfolio, consisting of all underlying stocks. The one-stock portfolio exhibits the highest level of exposure to the target characteristic, but zero diversification. Conversely, the all-stock portfolio displays no factor exposure in aggregate as the Z-scores have zero mean. However, the all-stock portfolio would have the highest level of diversification.

When considering multiple factor portfolios it seems clear that accommodating multiple factor exposures rather than just one must be at the expense of diversification. Further it would seem reasonable that factor correlation must play a role, particularly in the case of negative correlation. Strongly negatively correlated factors must, in some sense, be in danger of "cancelling one another out".

We shall see that these intuitions are true, but to varying degrees when we contrast the exposure and diversification properties of the composite portfolios formed from Characteristic Baskets with multiple tilted portfolios based on the Cumulative Normal.

5.1 One Factor Case

In this Subsection we calculate the exposure and concentration properties of single factor portfolios arising from a Characteristic Basket comprising of the top percentile of factor values and those for a tilt of power n using $S(x)$. This latter portfolio we will refer to as the "Exponential-Tilt" portfolio.

First consider the Characteristic Basket portfolio. Substituting $H(x)$ into (18) gives us the following expression for the concentration:

$$D\big[W^T[H(x)]\big] = \frac{1}{I[H(x)]} = \frac{1}{1-p} \tag{27}$$

This is precisely what one would expect, since we are restricting ourselves to factor values above the $100 * p^{th}$ percentile and equally weighting. For the exposure we get:

$$E_X\big[W^T[H(x)]\big] = \frac{e^{-\left[\mathrm{Erf}^{-1}(2p-1)\right]^2}}{\sqrt{2\pi}(1-p)} \tag{28}$$

Turning now to the Exponential-Tilt portfolio, note that from (18) we can write the single factor concentration as:

$$D\big[W^T[S(x)^n]\big] = \frac{I[S(x)^{2n}]}{I[S(x)^n]^2} = 1 + \frac{n^2}{(2n+1)} \tag{29}$$

where we have used the integral identity $I[S^n(x)] = 1/(n+1)$. For the exposure we get:

$$E_X[W^T[S^n(x)]] = \frac{I[S(x)^n * x]}{I[S(x)^n]} = (n+1) * I[S(x)^n * x] \tag{30}$$

where $I[S(x)^n * x]$ is easily obtained via numerical integration. For a "power-one" tilt ($n = 1$) we can evaluate this analytically yielding an exposure that is equal to $1/\sqrt{\pi} \approx 0.56$.

The two charts in Figure 3 illustrate the exposure and diversification trade-off: the greater the level of exposure, the less diversification there is. The left chart shows the exposure and Effective N of the Exponential-Tilt portfolio, resulting from varying the tilt power. As the tilt power increases the levels of exposure increase, whilst the Effective N decreases from 100% towards zero. The right chart shows the exposure and Effective N of a Characteristic Basket. Similarly, as we narrow our basket size the exposure increases and Effective N declines from 100% towards zero.

Figure 3: Exposure and Diversification of a Exponential-Tilt Portfolio (Left) and a Characteristic Basket (Right)
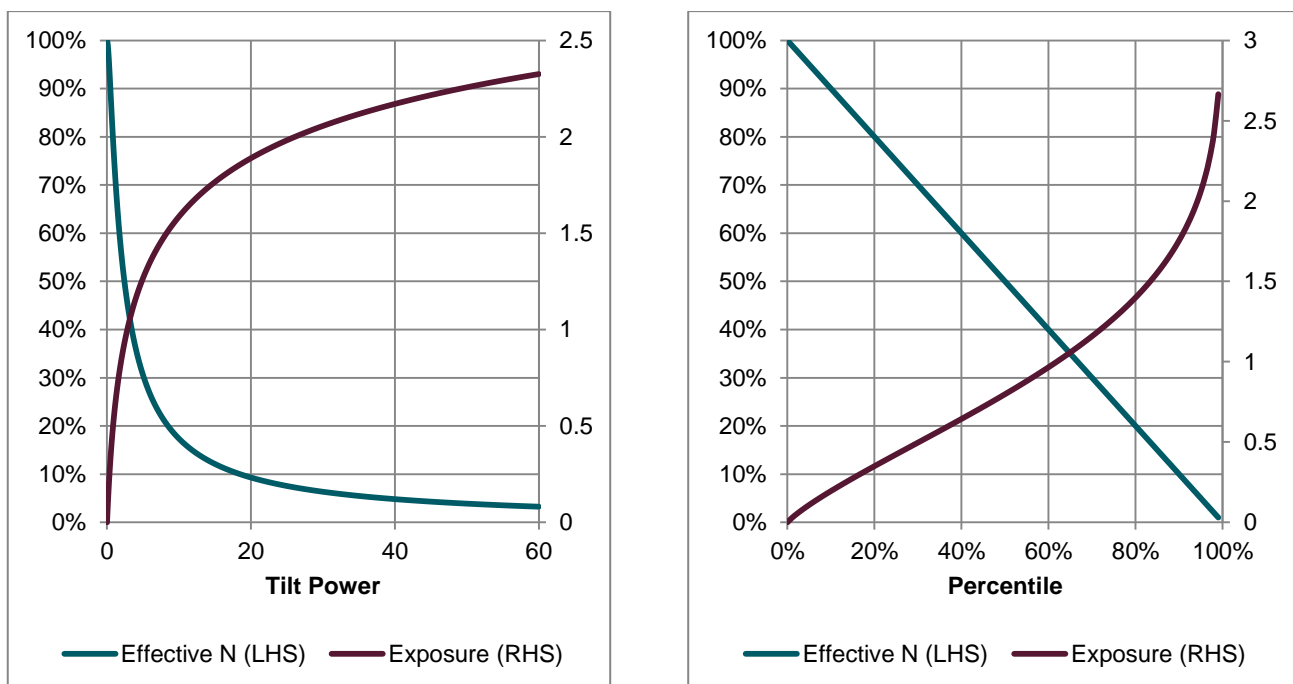
Given the trade-off between exposure and diversification for each of these constructions, an interesting and natural question is therefore: Which of the two methods delivers the highest degree of diversification for a given degree of exposure?

Figure 4 displays a plot of the Effective N verses exposure for each approach. For the Characteristic Basket we use the percentile (range 0 – 100%) to parameterize the Exposure/Effective N curve. For the Exponential-Tilt portfolio we use the tilt power (range 0 – 100) as a parameter.

Figure 4: Exposure verses Effective N for a Characteristic Basket and Exponential-Tilt
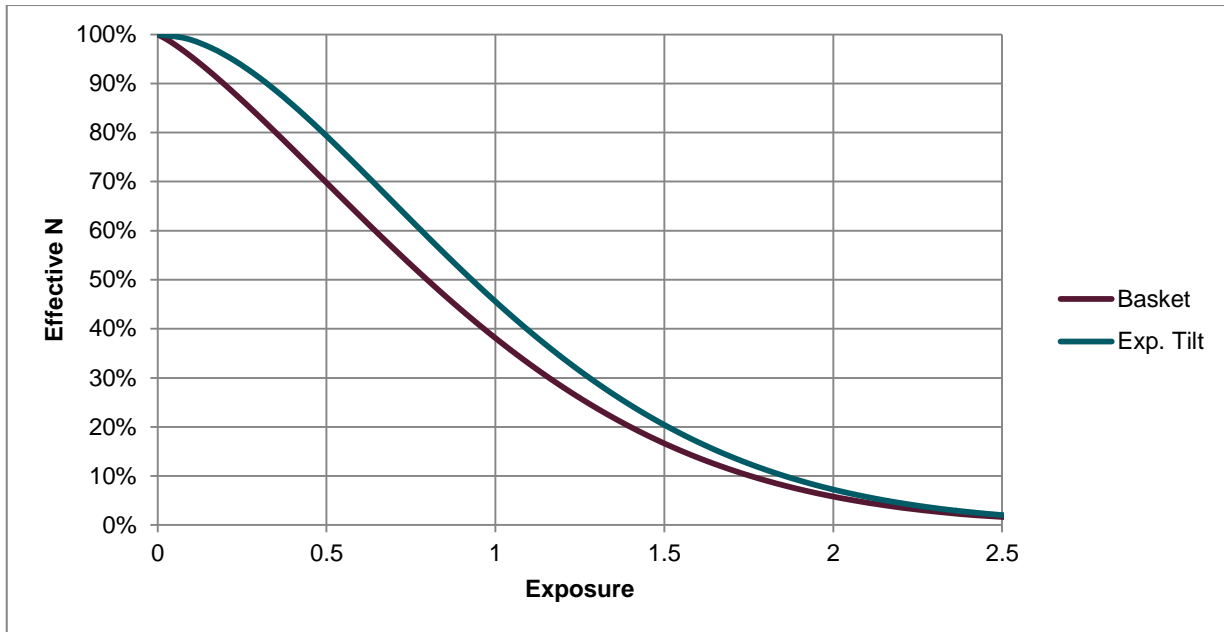
Clearly for any given degree of factor exposure, the Exponential-Tilt portfolio (based on powers of $S(x)$) always yields a more diverse portfolio than the Characteristic Basket. Or, conversely, for any given level of diversification, the Exponential-Tilt results in greater levels of exposure to the factor of interest than the Characteristic Basket approach.

Specifically, selecting the top half of the stock universe by factor score and then equal weighting to achieve the maximum possible the diversification of 50% is not as efficient as tilting using $S(x)$ raised to the power of 2.41, since this results in the same level of diversification but a 17% increase in factor exposure.

5.2 Two Factor Case

Extending the analysis to a two factor objective gives rise to our first multi-factor portfolio and allows us investigate the effect of factor correlation on our portfolio construction. We can repeat the above exercise comparing diversification and exposure profiles, but now allow for different degrees of correlation between factors. Note that in this multiple factor context, multiple tilting can be applied using either $S(x)$ or $H(x)$. The application of the latter will result in what is often described as the "intersection portfolio" [20] as it selects stocks that score above a threshold value for both factors.

In what follows we will refer to the portfolios arising from multiple tilts using various powers of $S(x)$ as "Multiple Tilt" portfolios. We will refer to the portfolios arising from multiple application of $H(x)$ with varying percentile levels as "Intersection" portfolios. Finally we will refer to composites of Characteristic Basket portfolios with varying percentile levels as "Composite Basket" portfolios.

We restrict our attention to the case where equal amounts of exposure for each of the two factors X and Y are required, that is for each weight function:

$$E_X[W] = E_Y[W]$$

(31)

In this situation, symmetry demands that the powers of both tilts are equal for the Multiple Tilt portfolio. For the Intersection portfolio and the Composite Basket, symmetry and simplicity imply that we shouldset each of the percentiles used, to the same level.

To get the expression for the exposure of the Multiple Tilt and Intersection portfolios we substitute (15) into (13) and simplify using (24). This gives:

$$E_X\big[W^T[F(x), F(y)]\big] = \frac{G[\rho, F(x) * x, F(y)]}{G[\rho, F(x), F(y)]} \tag{32}$$

Similarly inserting (15) into (14) and using (24) gives the concentration as:

$$D\big[W^T[F(x), F(y)]\big] = \frac{G[\rho, F(x)^2, F(y)^2]}{G[\rho, F(x), F(y)]^2} \tag{33}$$

where ρ is the correlation between factors. In these last two expressions we set $F(x) = S(x)^n$ for the Multiple Tilt and $F(x) = H(x)$ for the Intersection portfolio.

For the Composite Basket we substitute (28) into (22) and set $\alpha_j = 1/2$ to get the following expression for the exposure:

$$E_X[W^C[H(x), H(y)]] = \frac{(1+\rho)}{2} * \frac{e^{-[\mathrm{Erf}^{-1}(2p-1)]^2}}{\sqrt{2\pi}(1-p)} \tag{34}$$

Substituting (27) into (23) gives the following expression for the concentration:

$$D[W^C[H(x), H(y)]] = \frac{1}{2(1-p)} + \frac{1}{(1-p)^2} * G[\rho, H(x), H(y)] \tag{35}$$

Equations (32) – (35) can be evaluated using numerical integration for the various G – integrals. Therefore for each of our portfolios we can obtain expressions for exposure and concentration, for any chosen degree of factor correlation, which are parameterized by either tilt power or percentile.

We can now show the Exposure verses Effective N profiles introduced in the single factor case for any degree of correlation. To demonstrate the relationship, it is sufficient to show profiles for cases of positive, zero and negative correlation. Figures 5, 6 and 7 show them for representative correlations of +0.5, 0.0 and -0.5.

Figure 5: Exposure vs Effective N for the Composite Basket, Intersection and Multiple Tilt: Correlation = + 0.5
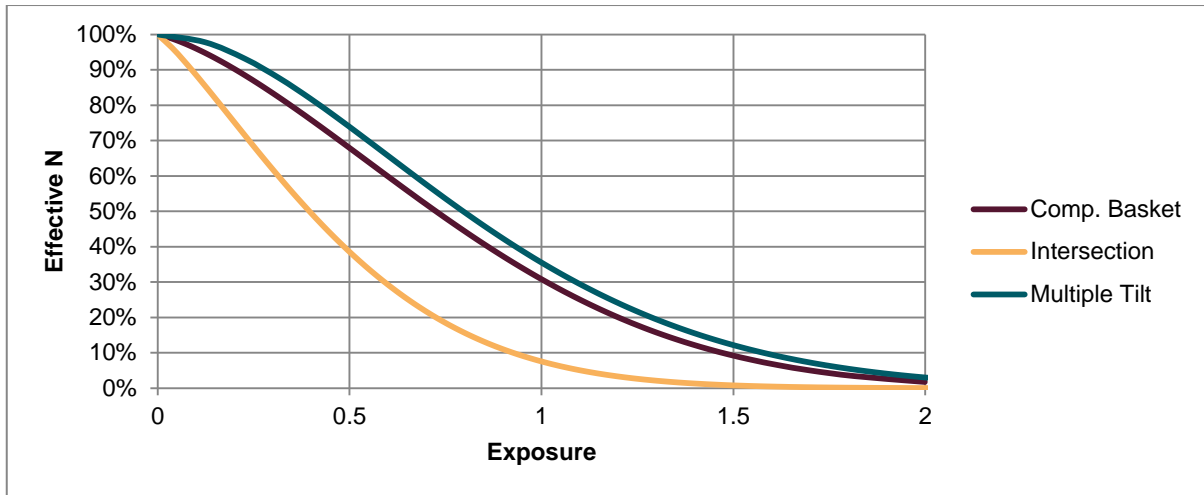
Figure 6: Exposure vs Effective N for the Composite Basket, Intersection and Multiple Tilt: Correlation = 0.0
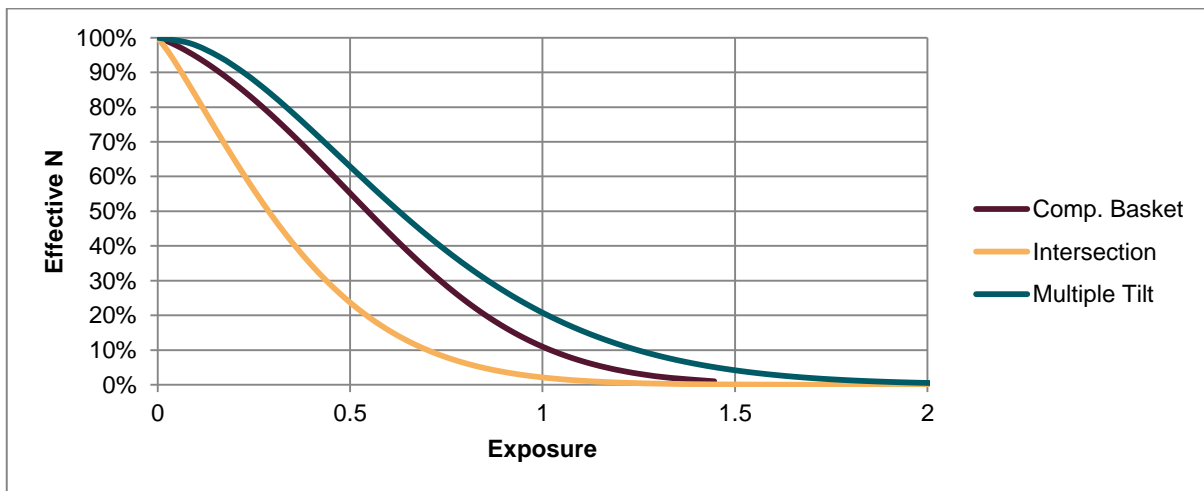
Figure 7: Exposure vs Effective N for the Composite Basket, Intersection and Multiple Tilt: Correlation = - 0.5
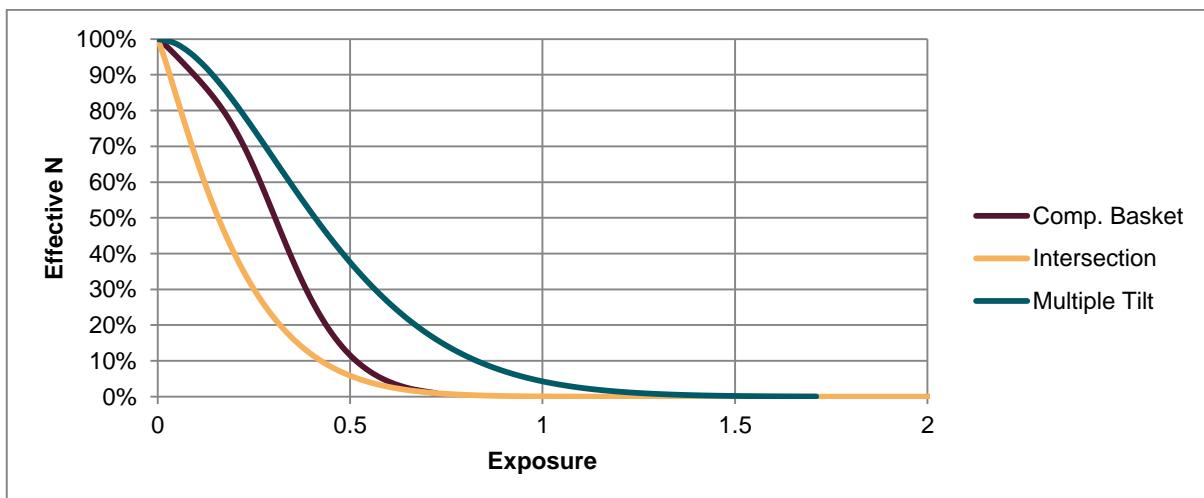
Firstly, note that the Multiple Tilt portfolio always results in a higher Effective N for a given degree of exposure than either of the two alternatives at all levels of correlation. The Intersection portfolio is the worst behaving, always being the least diversified at any level of correlation. The superiority of the multiple tilt exposure/diversification profile becomes more significant as factor correlation is increasingly negative. In particular, at a correlation of -0.5 and targeting an Effective N of 50%, a Composite Basket

(with percentiles = 73%) displays exposure of 0.3 compared to a Multiple Tilt (with tilt power = 1.3) exposure of 0.4.

These considerations lead to the conclusion that multiple tilting is a superior method of constructing a two factor portfolio than a composite of single factor baskets but this conclusion is sensitive to the choice of tilt function employed. The step function is "too rigid" to use in a multi-tilt context, rapidly narrowing the selection universe and consequently diversification levels to achieve higher levels of exposure. The use of the cumulative normal can be viewed as a "smeared intersection" that avoids these concentration problems whilst providing increased levels of factor exposure.

5.3 Three Factor Case

It would appear that the dimensionality of the three factor case makes it difficult to repeat the analysis of the previous two sections. This is because we now have three correlations to consider, whereas consideration of only one correlation was required in the two factor case. Matters are further complicated in that three different exposures are under consideration that all depend on the correlations, tilt powers and the selection percentiles associated with any given factor.

However, most cases of interest are covered by considering the following correlation permutations (+0.3,+0.3,+0.3), (+0.3,+0.3,-0.3), (+0.3,-0.3,-0.3) and (-0.3,-0.3,-0.3). Here we have chosen the magnitude of "0.3" to correspond roughly to the levels of factor (anti-) correlation one typically finds empirically. To simplify further we examine the case when tilt powers/percentiles are chosen for a given set of correlations that equalize the exposures calculated for each factor, that is:

$$E_X[W] = E_Y[W] = E_Z[W]$$

(36)

for each of our weighting functions and factors X, Y and Z.

Figures 8, 9, 10 and 11 show the Exposure/Effective N profiles for each set of correlations.

Figure 8: Correlations +0.3, +0.3 and +0.3
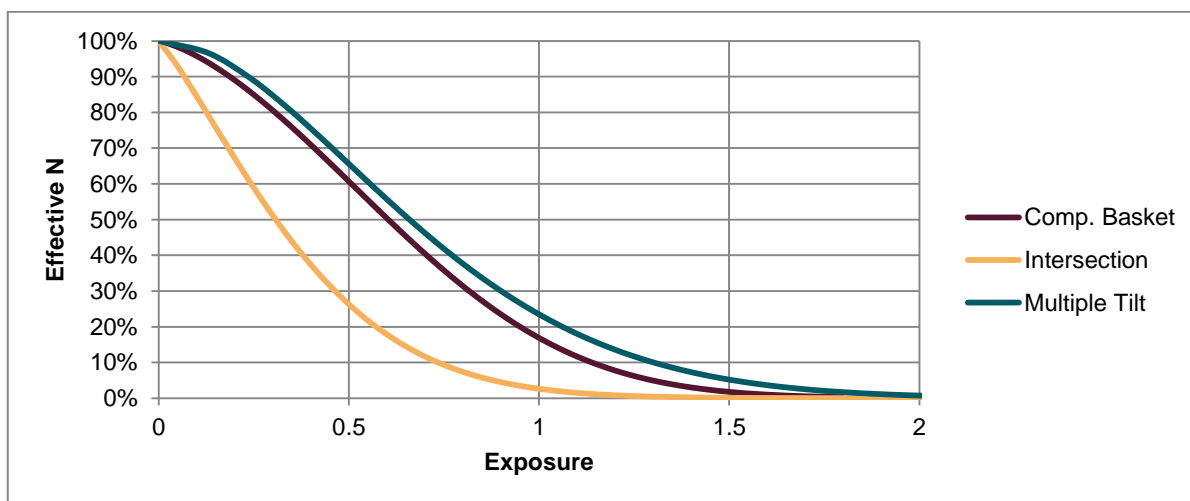
Figure 9: Correlations +0.3, +0.3 and -0.3
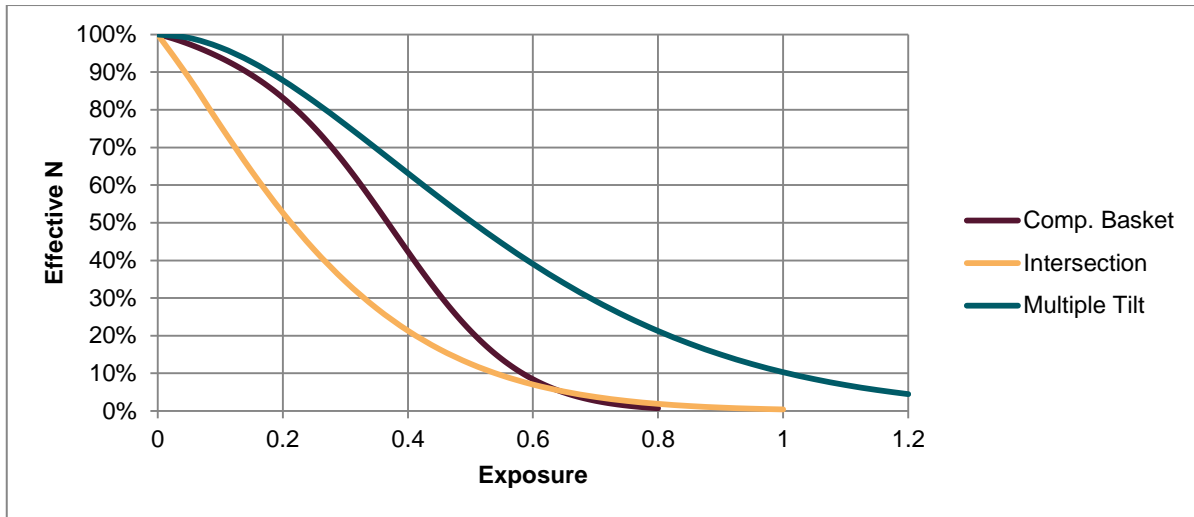
Figure 10: Correlations +0.3, -0.3 and -0.3
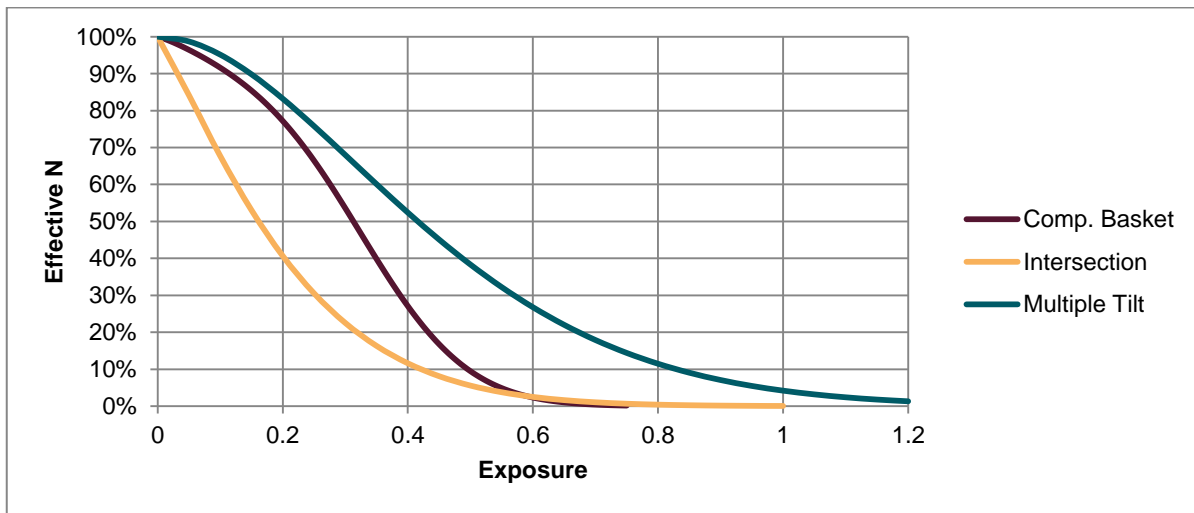
Figure 11: Correlations -0.3, -0.3 and -0.3
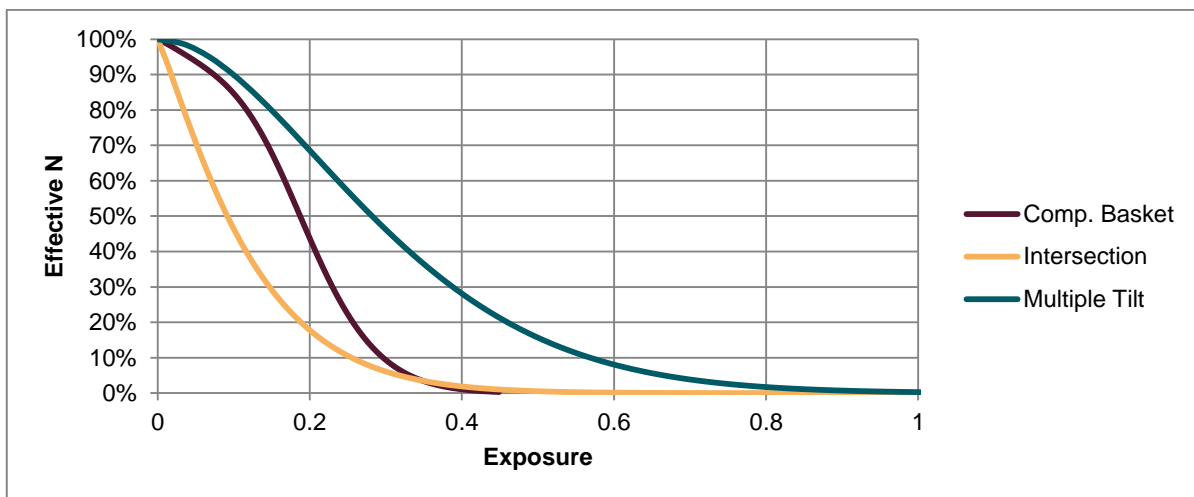
Clearly for all the sample sets of correlations, the Multiple Tilt provides a higher degree of diversification for any given level of exposure than does the Composite Basket or the Intersection portfolio. The differences become increasingly marked as the number of negative correlations increases. For example let's suppose that we wish to target the same level of exposure that is found in the single factor power-

one Exponential-Tilt portfolio introduced in Section 5.1 (i.e. $1/\sqrt{\pi}$). The resulting level of diversification for each correlation combination and construction technique is given in Table 1.

Table 1: Effective N for Equivalent Single Factor Power - One Exponential -Tilt Exposure

Correlation Combination	Effective N		
	Composite Basket	Intersection	Multiple Tilt
+0.3, +0.3, +0.3	54.05%	20.59%	59.21%
+0.3, +0.3, - 0.3	12.06%	8.70%	42.97%
+0.3, - 0.3, - 0.3	4.00%	3.36%	30.61%
- 0.3, - 0.3, - 0.3	0.01%	0.22%	10.31%

Only the Multiple Tilt shows good levels of diversification for each correlation combination. As a real-world example of the second correlation combination, consider the case of a multiple factor portfolio targeting quality, low volatility and value. Typically one finds that quality and value is the negatively correlated pair. Here the difference in diversification levels between the Multiple Tilt and the Composite Basket or Intersection portfolio is already quite dramatic. Indeed for the triple negative combination of correlations, levels of diversification are essentially zero for the Composite Basket and Intersection portfolio.

5.4 N-Factor Case

In the N factor case we have expressions for the exposure and concentration of the composite portfolio given by (22) and (23). Unfortunately we do not have similar simple expressions for the multiple tilt. Indeed since the number of correlations is equal to $N(N-1)/2$, the calculation of the various exposures for a multiple tilt becomes a complex process. To simplify matters let's assume that the N factors are uncorrelated. Also, as in the case of two and three factors, let's further simplify the problem by demanding that each factor has the same exposure and that this exposure arises from an identical tilt function, i.e. $F_i(x) = F(x)$.

In the case of a composite portfolio, setting $\rho_{ij} = 0$ for all $i \neq j$ and $\alpha_j = 1/N$ in (22) gives the following simple expression for the exposure:

$$E_{X_i}[W^C[F(x_1),..,F(x_N)]\,] = \frac{1}{N}E_X\big[W^T[F(x)]\big] = \frac{1}{N}\frac{I[F(x)*x]}{I[F(x)]} \tag{37}$$

In other words, the composite portfolio has an Nth of the exposure to factor X_i that it would have had in a single factor portfolio. This is the "averaging factor exposure" effect observed in composite portfolio construction approaches.

For the levels of concentration we obtain from (23):

$$D[W^C[F(x_1),..,F(x_N)]\,] = \frac{1}{N}D\big[W^T[F(x)]\big] + \frac{(N-1)}{N} = \frac{1}{N}\frac{I[F^2(x)]}{I[F(x)]^2} + \frac{(N-1)}{N} \tag{38}$$

That is the concentration is an Nth of the single factor level of concentration plus $(N-1)/N$.

To get the exposure in the case of a multiple tilt we substitute (15) with tilt functions $F(x)$ into (13). It relatively easy to show that:

$$E_{X_i}\big[W^T[F(x_1),..,F(x_N)]\big] = E_X\big[W^T[F(x)]\big] = \frac{I[F(x)*x]}{I[F(x)]} \tag{39}$$

In other words the level of exposure of each factor in a multifactor context is same as in the single factor case. It is similarly straight forward to show that the level of concentration is given by:

$$D\big[W^T[F(x_1),..,F(x_N)]\big] = D\big[W^T[F(x)]\big]^N = \left\{\frac{I[F^2(x)]}{I[F(x)]^2}\right\}^N \tag{40}$$

That is the concentration is the product of the N single factor levels of concentration.

Using these formulae we are now in the position to compare the exposure and concentration outcomes of an N factor multiple tilt portfolio based on the cumulative normal (Multiple Tilt portfolio) verses N single factor baskets used to form a composite factor portfolio (Composite Basket portfolio). Similar results can be obtained for the Intersection portfolio, but they are significantly poorer than for the other two portfolios in terms of lower exposure and more concentrated outcomes.

First let's examine how the exposure of these multi-factor portfolios vary as a function of the number of factors when we fix the Effective N equal to N_{eff}.

For the Multiple Tilt portfolio we set (40) equal to $1/N_{eff}$ to get:

$$D\big[W^T[S(x_1)^n,..,S(x_N)^n]\big] = \left\{\frac{I[S(x)^{2n}]}{I[S(x)^n]^2}\right\}^N = 1/N_{eff} \tag{41}$$

Using the integral identity $I[S(x)^n] = 1/(n+1)$ and solving the resulting equation for the tilt power n, gives:

$$n = \left(\frac{1}{N_{eff}}\right)^{1/N} - 1 + \sqrt{\left(\frac{1}{N_{eff}}\right)^{1/N}\left[\left(\frac{1}{N_{eff}}\right)^{1/N} - 1\right]} \tag{42}$$

We can now substitute this tilt power into our expression (39) for exposure:

$$E_{X_i}\big[W^T[S(x_1)^n,..,S(x_N)^n]\big] = (n+1)*I[S(x)^n * x] \tag{43}$$

Similarly for the Composite Basket portfolio we solve:

$$D[W^C[H(x_1),..,H(x_N)]] = \frac{1}{N*(1-p)} + \frac{(N-1)}{N} = 1/N_{eff} \tag{44}$$

for the percentile p, which gives:

$$p = \frac{1 - N_{eff}}{1 + N_{eff}(1/N - 1)} \tag{45}$$

We can then substitute this into the expression (37) for the exposure:

$$E_{X_i}[W^C[H(x_1),..,H(x_N)]] = \frac{e^{-\left[\mathrm{Erf}^{-1}(2p-1)\right]^2}}{\sqrt{2\pi}(1-p)\,N} \tag{46}$$

It is now possible to examine the cases where the requirement is that our N - factor portfolios have an Effective N of 10%, 25% and 50%. Figure 12 shows a plot of the ratio of the exposure of the Multiple Tilt to that of the Composite Basket verses number of factors.
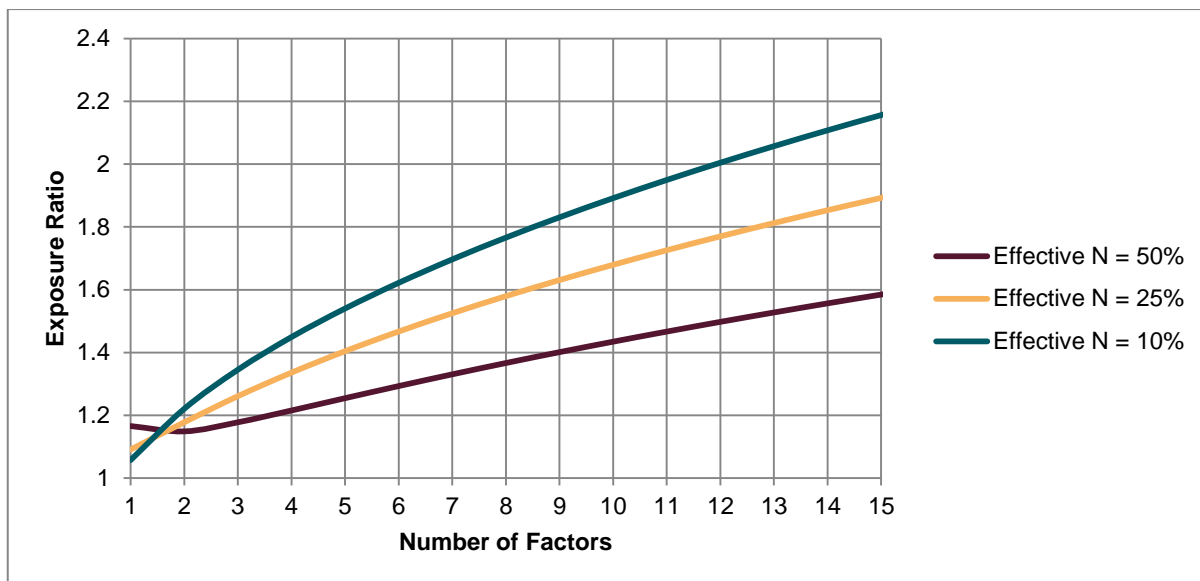
Clearly, for each level of diversification, the exposure for the Multiple Tilt portfolio is always greater than that of the Composite Basket since the ratio is always greater than one. Note also that, with the exception of 50% Effective N moving from one to two factors, as the number of factors increases the ratio of the exposures increases.

Alternatively, we can examine how diversification varies with the number of factors whilst fixing exposure at the same level as is found in a single factor power-one Exponential-Tilt $(1/\sqrt{\pi})$.

We already know from (39) that the Multiple Tilt portfolio's exposure is the same as the single factor exposure and that from (29) and (40), the concentration is:

$$D\big[W^T[S(x_1),..,S(x_N)]\big] = \left\{\frac{I[S^2(x)]}{I[S(x)]^2}\right\}^N = \left(\frac{4}{3}\right)^N \tag{47}$$

Matters are more complicated for the Composite Basket portfolio, as we must solve:

$$E_{X_i}\big[W^C[H(x_1),..,H(x_N)]\,\big] = \frac{e^{-\left[\mathrm{Erf}^{-1}(2p-1)\right]^2}}{\sqrt{2\pi}(1-p)\,N} = \frac{1}{\sqrt{\pi}} \tag{48}$$

for our percentile level p. This is relatively easy to do numerically. Given a solution we can then substitute it into (38) to obtain the level of concentration. The results are shown in Figure 13.

Figure 13: Effective N verses Number of Factors for the Composite Basket and the Multiple Tilt
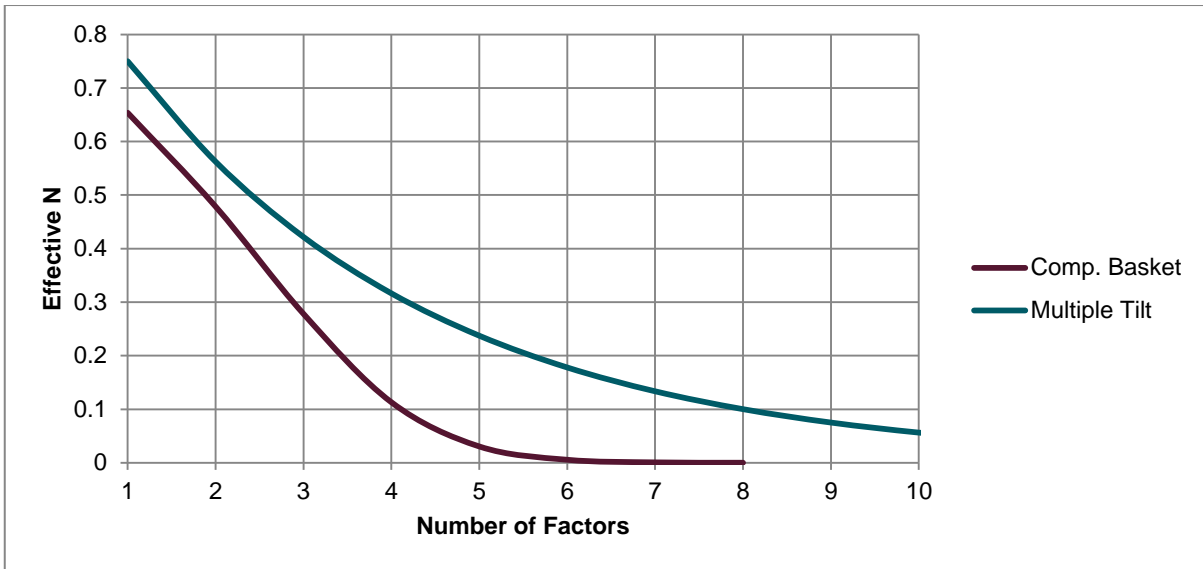
Clearly, for the same degree of exposure as in a single factor tilt portfolio for each of our N factors, the Multiple Tilt approach always results in a greater Effective N than the Composite Basket technique. Further, as the number of factors increases the percentage difference in the diversification becomes greater. For one factor it is about 15% but for five it is 679%. Indeed the Effective N of the Composite Basket tends to zero rapidly when the number of factors is greater than five. Recall that the diversification of Multiple tilt portfolio declines more gently as $(3/4)^N$.

6 Conclusion

We have shown how the concept of tilting towards or away from a factor encapsulates a number of portfolio construction techniques. In particular we have shown that the choice of the step function as the tilt function yields a Characteristic Basket, consisting of a given proportion of the universe by factor value. This is a commonly employed method of creating a single factor portfolio.

We have also shown that using the Cumulative Normal as the tilt function has several advantages over the step function. First in the single factor context the Cumulative Normal gives rise to factor portfolios that always have a greater degree of diversification for a given degree of exposure than those based on a Characteristic Basket.

In the case of multiple factors, factor correlation becomes important. We have shown that in the case of two and three factors, that multiple tilting using the Cumulative Normal again results in more diversified outcomes for a given degree of exposure than a composite of single factor Characteristic Baskets. Further we have shown that multiple tilting using the step function yields the Intersection portfolio which results in even greater levels of concentration. This superiority of employing a combination of the Cumulative Normal and multiple tilting becomes more pronounced as factor correlations become increasingly negative.

In the case of N factors, we have shown that, when the factors are more or less independent (zero correlation), for a target diversification corresponding to 10%, 25% and 50% (or indeed any other percentage) of the underlying equally weighted universe, multiple tilting using the Cumulative Normal again results in greater factor exposure than a composite of single factor Characteristic Baskets.

Finally we have shown that, if we have a certain amount of target exposure in mind for each of our N factors, say the amount that would result from a single factor portfolio tilt, the degree of diversification is always higher for the multiple tilt employing the Cumulative Normal than for a composite of Characteristic Baskets. The percentage difference in diversification becomes greater as the number of factors increases. Indeed beyond five factors the composite Characteristic Basket yields an Effective N that rapidly tends to zero.

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